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THE AVERAGE OF AN ANALYTIC FUNCTIONAL¹

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1. Conceive a particle free to wander along the x -axis. Suppose the probability that it wander a given distance independent

(1) of the position from which it starts to wander,

(2) of the time when it starts to wander,

(3) of the direction in which it wanders.

It may be shown² that under these circumstances, the probability that after a time, t , it has wandered from the origin to a position lying between $x=x_0$ and $x=x_1$ is

$$\frac{1}{\sqrt{\pi ct}} \int_{x_0}^{x_1} e^{-\frac{x^2}{4t}} dx$$

where t is the time and c is a certain constant which we can reduce to 1 by a proper choice of units. This choice we shall make in what follows. The exponential form of this integral needs no comment, while the mode in which t enters results from the fact that

$$\frac{1}{\sqrt{\pi(t_1+t_2)}} \int_{x_0}^{x_1} e^{-\frac{x^2}{4(t_1+t_2)}} dx = \frac{1}{\sqrt{\pi t_1}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{\pi t_2}} \int_{x_0}^{x_1} e^{-\frac{(y-x)^2}{4t_2}} dy \right] e^{-\frac{x^2}{4t_1}} dx$$

This identity will be presupposed in all that follows.

Let us now consider a particle wandering from the origin for a given period of time, say from $t=0$ to $t=1$. Its position will then be a function of the time, say $x=f(t)$. There are certain quantities—functionals—which may depend on the whole set of values of f from $t=0$ to $t=1$. If we take a large number of particles (i.e. a large number of values of f) at random, it is natural to suppose that the average value of the functional will often approach a limit, which we may call the average value of the functional over its entire range. What will this average be, and how shall we find it?

If $F\{f\}$ is a functional depending on the values of f for only a finite num-

ber of values of the argument of f —if $F\{f\}$ is a function³ of $f(t_1), f(t_2), \dots, f(t_n)$ of the form $\Phi[f(t_1), \dots, f(t_n)]$ —then it is easy enough to give a natural definition of the average of F , which we shall write $A\{F\}$. We can reasonably say

$$A\{F\} = \frac{1}{\sqrt{\pi^n t_1(t_2 - t_1) \dots (t_n - t_{n-1})}} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \Phi(x_1, \dots, x_n) e^{-\frac{x_1^2}{t_1} - \frac{(x_2 - x_1)^2}{t_2 - t_1} - \dots - \frac{(x_n - x_{n-1})^2}{t_n - t_{n-1}}} dx_1 \dots dx_n.$$

In particular if $F\{f\} = [f(t_1)]^{m_1} [f(t_2)]^{m_2} \dots [f(t_n)]^{m_n}$, then

$$\begin{aligned} A\{F\} &= \frac{1}{\sqrt{\pi^n t_1(t_2 - t_1) \dots (t_n - t_{n-1})}} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} x_1^{m_1} \dots x_n^{m_n} e^{-\frac{x_1^2}{t_1} - \dots - \frac{(x_n - x_{n-1})^2}{t_n - t_{n-1}}} dx_1 \dots dx_n \\ &= \frac{1}{\sqrt{\pi^n t_1(t_2 - t_1) \dots (t_n - t_{n-1})}} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} y_1^{m_1} (y_2 + y_1)^{m_2} \dots (y_n + y_{n-1} + \dots + y_1)^{m_n} e^{-\frac{y_1^2}{t_1} - \dots - \frac{y_n^2}{t_n - t_{n-1}}} dy_1 \dots dy_n \\ &= \frac{1}{\sqrt{\pi^n}} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} (t_1^{1/2} z_1)^{m_1} [(t_2 - t_1)^{1/2} z_2 + t_1^{1/2} z_1]^{m_2} \dots [(t_n - t_{n-1})^{1/2} z_n + \dots + t_1^{1/2} z_1]^{m_n} e^{-z_1^2 - \dots - z_n^2} dz_1 \dots dz_n. \end{aligned}$$

This latter integral is in the form

$$\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} P(z_1, \dots, z_n) e^{-z_1^2 - \dots - z_n^2} dz_1, \dots, dz_n$$

where P is a polynomial, and can be evaluated by means of the well known formulae:

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-y^2} y^{2n+1} dy &= 0, \\ \int_{-\infty}^{\infty} e^{-y^2} y^{2n} dy &= \sqrt{\pi} \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2^n}. \end{aligned}$$

We can thus easily evaluate $A\{F\}$ as a polynomial in t_1, t_2, \dots, t_n , which we shall call $P_{m_1, \dots, m_n}(t_1, \dots, t_n)$. It is easy to show that if $\sum m_k$ is odd,

$$P_{m_1, \dots, m_n}(t_1, \dots, t_n) = 0.$$

To return to the more general functional: there is a large class of so-called analytic functionals,⁴ which may be expanded in the form of series such as

$$F\{f\} = a_0 + \int_0^1 f(x)\varphi_1(x)dx + \int_0^1 \int_0^1 f(x)f(y)\varphi_2(x,y)dxdy + \dots$$

$$+ \int_0^1 \dots \int_0^1 f(x_1) \dots f(x_n)\varphi_n(x_1, \dots, x_n) dx_1 \dots dx_n + \dots$$

and an even wider class of what may be called Stieltjes analytic functionals, in which the general term

$$\int_0^1 \dots \int_0^1 f(x_1) \dots f(x_n)\varphi_n(x_1, \dots, x_n)dx_1, \dots dx_n$$

is replaced by the Stieltjes integral⁶

$$\int_0^1 \dots \int_0^1 f(x_1) \dots f(x_n)d\psi_n(x_1, \dots, x_n).$$

In what follows, we shall confine our discussion to Stieltjes analytic functionals, which we shall call simply analytic. The problem with which we are now concerned is the definition of the average of an analytic functional. Now, the first property which any average ought to fulfil is that the average of the sum of two functionals should equal the sum of their averages. We should expect, therefore, that:

(a) Over a wide range of cases, the average of a series should equal the series of the averages of the terms;

(b) The average of a Stieltjes integral, single or multiple, of a given functional with respect to such parameters as it may contain, should be equal to the integral of the average;

(c) A constant multiplied by the average of a functional should equal the average of a constant times the functional.

In accordance with this, we get the following natural definition of the average of the analytic functional F .

$$A\{F\} = A\left\{a_0 + \int_0^1 f(x)d\psi_1(x) + \int_0^1 \int_0^1 f(x)f(y)d\psi_2(x,y) + \dots\right\}$$

$$= a_0 + A\left\{\int_0^1 f(x)d\psi_1(x)\right\} + A\left\{\int_0^1 \int_0^1 f(x)f(y)d\psi_2(x,y)\right\} + \dots$$

$$= a_0 + \int_0^1 A\{f(x)\}d\psi_1(x) + \int_0^1 \int_0^1 A\{f(x)f(y)\}d\psi_2(x,y) + \dots$$

We have already seen how to determine $A\{f(x_1)\dots f(x_n)\}$ as a polynomial in the x_k 's. Hence whenever the above series converges, we have now a way of obtaining a perfectly definite value for $A\{F\}$. It may be noted that every term in the above expression in which the sign of integration is repeated an odd number of times is identically zero.

If $A\{F\}$ is to behave as we should expect it to behave, there are certain properties which it must fulfil, at least over a large and important class of cases. Among these are the following:

$$(1) \quad A\{F_1\} + A\{F_2\} = A\{F_1 + F_2\}$$

$$(2) \quad cA\{F_1\} = A\{cF_1\}$$

$$(3) \quad \sum_1^{\infty} A\{F_n\} = A\left\{\sum_1^{\infty} F_n\right\}$$

(4) If F_x is a functional depending on the parameter x , and $u(x)$ is a function of limited total variation, then

$$\int_a^b A\{F_x\} du = A\left\{\int_a^b F_x du\right\}$$

(5) Suppose $F_{t_1, \dots, t_n}(x_1, \dots, x_n)$ be defined as $F\left\{f_{t_1, \dots, t_n}^{x_1, \dots, x_n}(t)\right\}$, where

$f_{t_1, \dots, t_n}^{x_1, \dots, x_n}(t)$ assumes the value

$$x_k + \frac{(t - t_k)(x_{k+1} - x_k)}{t_{k+1} - t_k}$$

for $t_k \leq t \leq t_{k+1}$. Then

$$A\{F\} = \lim \pi^{-\frac{n}{2}} \prod_1^n (t_k - t_{k-1})^{-1/2} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} F_{t_1, \dots, t_n}(x_1, \dots, x_n) e^{-\frac{x_1^2}{t_1} - \sum_2^n \frac{(x_k - x_{k-1})^2}{(t_k - t_{k-1})}} dx_1 \dots dx_n,$$

where the limit is taken as the t_k 's increase in number in such a manner to divide the interval from 0 to 1 more and more finely.

2. The next task before us is to investigate hypotheses which are sufficient to guarantee the validity of propositions (1)–(5). Propositions (1) and (2) require indeed very little discussion, for they are always satisfied when the series for $F_1, F_2, A\{F_1\}$ and $A\{F_2\}$ converge. In (3), let F_1, \dots, F_n, \dots , and the series $\sum F_n$ all possess averages, and let

$$\sum_1^{\infty} F_n = \sum_1^m F_n + R_m$$

where $A\{R_m\}$ vanishes as m increases without limit. Then

$$A\left\{\sum_1^{\infty} F_n\right\} = \sum_1^m A\{F_n\} + A\{R_m\}.$$

Therefore

$$\lim_{m \rightarrow \infty} \left| A\left\{\sum_1^{\infty} F_n\right\} - \sum_1^m A\{F_n\} \right| = 0$$

and (3) is proved. If $\sum F_n$ converges and $\lim A\{R_m\} = 0$, we shall say $\sum F_n$ converges *smoothly*.

Proposition (4) reduces to the ordinary inversion of a multiple Stieltjes integral when $F_x\{f\}$ is of the form

$$\int_0^1 \dots \int_0^1 f(x_1) \dots f(x_n) d\psi_n(x_1, \dots, x_n, x)$$

and ψ_n is a function of limited variation in x_1, \dots, x_n . What we wish to prove is that

$$\int_a^b \left\{ \int_0^1 \dots \int_0^1 A \{f(x_1) \dots f(x_n)\} d\psi_n(x_1, \dots, x_n) \right\} du \\ = A \left\{ \int_a^b \left\{ \int_0^1 \dots \int_0^1 f(x_1) \dots f(x_n) d\psi_n(x_1, \dots, x_n) \right\} du \right\}$$

Now

$$\int_a^b \left\{ \int_0^1 \dots \int_0^1 f(x_1) \dots f(x_n) d\psi_n(x_1, \dots, x_n) \right\} du = \\ \int_a^b \left[\lim_{\alpha, \dots, \vartheta} \sum f(\xi_{1\alpha}) \dots f(\xi_{n\vartheta}) \Delta_{x_1 \alpha \dots x_n \vartheta}^{x_1, \alpha+1 \dots x_n, \vartheta+1} \psi_n(x_1, \dots, x_n, x) \right] du.$$

If in this latter expression the total variation of ψ_n is less than a quantity independent of x , we can permute the \int_a^b and the \lim , and get

$$\lim_{\alpha, \dots, \vartheta} \sum f(\xi_{1\alpha}) \dots f(\xi_{n\vartheta}) \Delta_{x_1 \alpha \dots x_n \vartheta}^{x_1, \alpha+1 \dots x_n, \vartheta+1} \int_a^b \psi_n(x_1, \dots, x_n, x) du,$$

which we may write

$$\int_0^1 \dots \int_0^1 f(x_1) \dots f(x_n) d \left[\int_a^b \psi_n(x_1, \dots, x_n, x) du \right].$$

In this we suppose f uniformly continuous. It is easy to show that on our assumptions $\int_a^b \psi_n(x_1, \dots, x_n, x) du$ is of limited variation. Consequently we obtain

$$A \left\{ \int_a^b \left\{ \int_0^1 \dots \int_0^1 f(x_1) \dots f(x_n) d\psi_n(x_1, \dots, x_n, x) \right\} du \right\} \\ = A \left\{ \int_0^1 \dots \int_0^1 f(x_1) \dots f(x_n) d \left[\int_a^b \psi_n(x_1, \dots, x_n, x) du \right] \right\} \\ = \int_0^1 \dots \int_0^1 A \{f(x_1) \dots f(x_n)\} d \left[\int_a^b \psi_n(x_1, \dots, x_n, x) du \right].$$

A further transformation just like the preceding turns this into

$$\int_a^b \left\{ \int_0^1 \dots \int_0^1 A \{f(x_1) \dots f(x_n)\} d\psi_n(x_1, \dots, x_n, x) \right\} du$$

so that we have now a sufficient condition for the validity of our theorem. The extension to non-homogeneous terminating analytic functionals is obvious. The extension to non-terminating analytic functionals may be deduced with the help of (3) and a well-known theorem on the integration of uniformly convergent series, and reads as follows: let F_x be an analytic functional of the form

$$a_0 + \sum_1^\infty \int_0^1 \dots \int_0^1 f(x_1) \dots f(x_n) d\psi_n(x_1, \dots, x_n, x)$$

where the total variation of each ψ_n is less than some quantity independent of x , and let each ψ_n be uniformly continuous in x over the interval (a, b) . Let $u(x)$ be a function of limited total variation in x over the same interval. Let $A\{\int_a^b F_x du\}$ exist, and let

$$\lim_{m \rightarrow \infty} \sum_{n=m}^{\infty} \int_0^1 \dots \int_0^1 A\{f(x_1) \dots f(x_n)\} d\psi_n(x_1, \dots, x_n, x) = 0$$

uniformly in x . Then

$$\int_a^b A\{F_x\} du = A\left\{\int_a^b F_x du\right\}.$$

As to (5), let us begin as above with a functional of the form

$$\Phi\{f\} = \int_0^1 \dots \int_0^1 f(x_1) \dots f(x_n) d\psi_n(x_1, \dots, x_n).$$

Consider

$$I = \int_0^1 \dots \int_0^1 A\{f(x_1) \dots f(x_n)\} d\psi_n(x_1, \dots, x_n),$$

where A is taken in the original sense as an n -fold integral. By definition

$$I = \lim_{\alpha, \dots, \theta} \sum A\{f(\xi_{1\alpha}) \dots f(\xi_{n\theta})\} \Delta_{x_{1\alpha} \dots x_{n\theta}}^{x_{1,\alpha+1} \dots x_{n,\theta+1}} \psi_n(x_1, \dots, x_n),$$

where $x_{k0}=0, x_{k1}, \dots, x_{k\mu}=1$ is an increasing sequence of numbers, ξ_{kK} lies between x_{kK} and $x_{k,K+1}$ and \lim is taken as $\max(x_{k,K+1}-x_{kK})$ approaches 0. Let V be the total variation of ψ_n as its arguments range from 0 to 1, and let M stand for $\max(x_{k,K+1}-x_{kK})$. Let Q stand for the least upper bound of the variation of $A\{f(x_1) \dots f(x_n)\}$ as the point (x_1, \dots, x_n) wanders over an interval $\left(\begin{smallmatrix} x_{1,\alpha+1} \dots x_{n,\theta+1} \\ x_{1\alpha} \dots x_{n\theta} \end{smallmatrix}\right)$. Then

$$(6) \quad \left| I - A\left\{ \sum_{\alpha, \dots, \theta} f(\xi_{1\alpha}) \dots f(\xi_{n\theta}) \Delta_{x_{1\alpha} \dots x_{n\theta}}^{x_{1,\alpha+1} \dots x_{n,\theta+1}} \psi_n(x_1, \dots, x_n) \right\} \right| \leq VQ.$$

Now, let $f_m(x)$ be that function whose graph is the broken line with corners at $(x_{(K)}, f(x_{(K)}))$, where $0 \leq K \leq \mu, x_{(0)}=0, x_{(\mu)}=1$. Then if x lies between $x_{(K)}$ and $x_{(K+1)}$, $f_m(x)$ is of the form $af_m(x_{(K)}) + bf_m(x_{(K+1)}) \div a+b$. It follows that if $(\xi_{1\alpha}, \dots, \xi_{n\theta})$ lies in the interval $\left(\begin{smallmatrix} x_{(\alpha+1)} \dots x_{(\theta+1)} \\ x_{(\alpha)} \dots x_{(\theta)} \end{smallmatrix}\right)$, $A\{f_m(\xi_{1\alpha}) \dots f_m(\xi_{n\theta})\}$ is of the form

$$\frac{a_1 C_1 + a_2 C_2 + \dots + a_p C_p}{a_1 + a_2 + \dots + a_p}$$

where each C_k is the value of some $A\{f(\xi_{1\alpha}) \dots f(\xi_{n\theta})\}$ such that $(\xi_{1\alpha}, \dots, \xi_{n\theta})$ is a corner of the hyperparallelopiped $\left(\begin{smallmatrix} x_{(\alpha+1)} \dots x_{(\theta+1)} \\ x_{(\alpha)} \dots x_{(\theta)} \end{smallmatrix}\right)$. It readily results from considerations of continuity that $A\{f_m(\xi_1) \dots f_m(\xi_p)\}$ is of the form $A\{f(\eta_{1\alpha}) \dots f(\eta_{n\theta})\}$, where $(\eta_{1\alpha}, \dots, \eta_{n\theta})$ also lies in the interval $\left(\begin{smallmatrix} x_{(\alpha+1)} \dots x_{(\theta+1)} \\ x_{(\alpha)} \dots x_{(\theta)} \end{smallmatrix}\right)$.

Making use of this fact, and of the fact that

$$\frac{1}{\sqrt{\pi^n(t_2 - t_1) \dots (t_n - t_{n-1})}} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \Phi(x_1, \dots, x_n) e^{-\frac{x_1^2}{t_1} - \dots - \frac{(x_n - x_{n-1})^2}{t_n - t_{n-1}}} dx_1 \dots dx_n$$

is an increasing functional of $\Phi(x_1, \dots, x_n)$, we can draw the conclusion that

$$A\{\Phi\{f_m\}\} = A\left\{\int_0^1 \dots \int_0^1 f_m(x_1) \dots f_m(x_n) d\psi_n(x_1, \dots, x_n)\right\}$$

if it exists, lies between the uppermost and lowermost values of

$$\sum_{\alpha, \dots, \vartheta} A\{f_m(\xi_{1\alpha}) \dots f_m(\xi_{n\vartheta})\} \Delta_{x(\alpha) \dots x(\vartheta)}^{x(\alpha+1) \dots x(\vartheta+1)} \psi_n(x_1, \dots, x_n)$$

and hence of

$$\sum_{\alpha, \dots, \vartheta} A\{f(\eta_{1\alpha}) \dots f(\eta_{n\vartheta})\} \Delta_{x(\alpha) \dots x(\vartheta)}^{x(\alpha+1) \dots x(\vartheta+1)} \psi_n(x_1, \dots, x_n)$$

From this and (6) we can deduce

$$(7) \quad |I - A\{\Phi\{f_m\}\}| \leq 2VQ$$

where Q is taken for $x_{kK} = x_{(K)}$.

This proves our theorem for homogeneous analytic functionals. In precise terms, then, our general theorem will read: let

$$F\{f\} = a_0 + \sum_1^\infty \int_0^1 \dots \int_0^1 f(x_1) \dots f(x_n) d\psi_n(x_1, \dots, x_n) = a_0 + \sum_1^\infty F_n\{f\}$$

be an analytic functional. Let V_n stand for the total variation of F_n as its arguments range from 0 to 1: Let $(x_{(\alpha)}, \dots, x_{(\mu)})$ be a set of numbers in ascending order from 0 to 1, inclusive. Let M stand for $\max(x_{(K+1)} - x_{(K)})$ and Q_n for the upper bound of the variation of $A\{f(x_1) \dots f(x_n)\}$ as the point (x_1, \dots, x_n) wanders over the interval $\left(\frac{x(\alpha+1) \dots x(\vartheta+1)}{x(\alpha) \dots x(\vartheta)}\right)$. Let $f_n(x)$ be the function whose graph is the broken line with corners at $(0, 0)$ and $(x_{(K)}, f(x_{(K)}))$. Then if

$$(a) \quad \lim_{M \rightarrow 0} \sum_1^\infty V_k Q_k = 0,$$

(b) $A\{F_n\{f_\mu\}\}$ exists for every μ and n according to the definition of A as a multiple integral;

(c) the series for F converges smoothly;

$$(d) \lim_{n \rightarrow \infty} A\left\{\sum_n^\infty F_m\{f_\mu\}\right\} \text{ exists for every } \mu, \text{ when } A \text{ is taken as a}$$

multiple integral; it follows that

$$A\{F\} = \lim_{M \rightarrow 0} A\{F\{f_m\}\},$$

where the first A is defined in the sense of the average of an analytic functional, and the second as a multiple integral. A precisely analogous theorem holds when $f_m(x)$ instead of a broken straight line is any broken line with corners at $(0, 0)$ and at $(x_{(K)}, f(x_{(K)}))$, and consists of monotone arcs between these points. This last theorem makes our average of a functional the limit of the average of a function of a discrete set of variables, and justifies our use of the term average.

¹ The problem of the mean of a functional has been attached by Gâteaux (*Bull. Soc. Math. de France*, 1919, pp. 47-70). The idea of using the analytic functional as a basis is there found. The actual definition, however, is essentially different, and does not lend itself readily to the treatment of the Brownian Movement, for which the present method is especially adapted.

² Einstein, *Leipzig, Annalen Physik*, 17, 905.

³ We here take $t_1 < t_2 < \dots < t_n$.

⁴ Cf. V. Volterra, *Fonctions des Lignes*.

⁵ Cf. P. J. Daniell, *Annals of Mathematics*, Sept., 1919, p. 30.

ON THE CALCULATION OF THE X-RAY ABSORPTION FREQUENCIES OF THE CHEMICAL ELEMENTS

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The K critical absorption frequency of a chemical element is the highest frequency of vibration known to be characteristic of that element. In our laboratory we have measured the K critical absorption frequencies of most of the chemical elements by the ionization method. This data, together with measurements made elsewhere by the photographic method, may be found in Table 2 of a report by the author on Data Relating to X-Ray Spectra, which has been published by the National Research Council.

At a symposium on Ultra-Violet Light and X-Rays, held at the meeting of the American Association for the Advancement of Science at St. Louis in December 1919,¹ I presented a set of computations of the K critical absorption frequencies based on the Rutherford-Bohr theory of atomic structure and the mechanism of radiation. The computed values equalled the observed values to within one or two per cent. In these computations the electrons were supposed to revolve in orbits which lay in planes passing through the nucleus of the atom.

Later I presented² to the National Academy of Sciences and to the American Physical Society computations of these K critical absorption frequencies, calculated on the assumption that the orbits did not all lie in planes through the nucleus. I assumed that the orbits were cir-